Taylor Series Expansions

a)

$n=0$ $\sin(u)$ $\quad$ $n=4$ $\sin(u)$

$n=1$ $\cos(u)$ $\quad$ $n=5$ $\cos(u)$

$n=2$ $-\sin(u)$ $\quad$ $n=6$ $-\sin(u)$

$n=3$ $-\cos(u)$ $\quad$ $n=7$ $-\cos(u)$

$$u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 + \dots$$

if $u = 2x$

$$2x - \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 - \frac{1}{7!}(2x)^7 + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{(2n+1)}}$$

b) $n=0$ $\ln(2x)$ $\quad$ $n=1$ $\frac{1}{x}$ $\quad$ $n=2$ $\frac{-1}{x^2}$ $\quad$ $n=3$ $\frac{2}{x^3}$ $\quad$ $n=4$ $\frac{-6}{x^4}$

$$\ln(2) + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{2\cdot 3}(x-1)^3 - \frac{6}{2\cdot 3\cdot 4}(x-1)^4 + \dots$$

$$= \boxed{\ln(2) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n}$$

c) $n=0$ e^2 $\quad$ $n=1$ $2e^2$ $\quad$ $n=2$ $4e^2$ $\quad$ $n=3$ $8e^2$

$$e^2 + \frac{2e^2}{1!}(x-1) + \frac{4e^2}{2!}(x-1)^2 + \dots = \boxed{\sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n}$$

d) $n=0$ $3x^2-2x+5$ $\quad$ $n=1$ $6x-2$ $\quad$ $n=2$ 6 $\quad$ $n=3$

5 $\quad$ 2 $\quad$ 6 $\quad$ 0

$$5 + \frac{-2(x)}{1!} + \frac{6x^2}{2!} = \boxed{3x^2 - 2x + 5}$$

e) $n=0 \Rightarrow 6$ $\quad$ $n=1 \Rightarrow 4$ $\quad$ $n=6 \Rightarrow 6$ $\quad$ $n=3 \Rightarrow 0$

$$6 + \frac{4(x-1)}{1!} + \frac{6(x-1)^2}{2!} + 0$$

$$= \boxed{6 + 4(x-1) + 3(x-1)^2}$$

f) $n=0$ $\frac{d}{dx} f(1) = \frac{1}{6}$ $n=1$ $\frac{d^2}{dx^2} f(1) = -\frac{1}{9}$ $n=2$ $\frac{d^2}{dx^2} f(1) = -\frac{1}{54}$

$n=3$ $\frac{d^3}{dx^3} f(1) = \frac{10}{27}$ $n=4$ $\frac{d^4}{dx^4} f(1) = \frac{-71}{81}$

$$\boxed{\frac{1}{6} - \frac{1}{9} \cdot (x-1) - \frac{1}{54} \cdot \frac{1}{2!} (x-1)^2 + \frac{10}{27} \cdot \frac{1}{3!} (x-1)^3 - \frac{71}{81} \cdot \frac{1}{4!} (x-1)^4 + \ldots}$$

g) $n=0$ $\cosh(x-3)$ $n=1$ $\sinh(x-3)$ $n=2$ $\cosh(x-3)$ $n=3$ $\sinh(x-3)$

$$\frac{e^{-2}+e^{2}}{2} + \frac{e^{-2}-e^{2}}{2\cdot 1!}(x-1) + \frac{e^{-2}+e^{2}}{2\cdot 2!}(x-1)^2 + \frac{e^{-2}-e^{2}}{2\cdot 3!}(x-1)^3 + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{e^{-2}+(-1)^n e^{2}}{2\cdot n!}(x-1)^n}$$

h) $$\boxed{\sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots}$$

i) $$\sum_{n=0}^{\infty} \frac{F^{(n)}(x)}{n!}(a-x)^n$$

$$\boxed{= f(x) + \frac{f'(x)}{1!}(a-x) + \frac{f''(x)}{2!}(a-x)^2 + \frac{f'''(x)}{3!}(a-x)^3 + \dots}$$

j) $$\sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!}(a+h-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(a+h-a) + \frac{f''(a)}{2!}(a+h-a)^2 + \frac{f'''(a)}{3!}(a+h-a)^3 + \dots$$

$$\boxed{= f(a) + \frac{f'(a)}{1!}(h) + \frac{f''(a)}{2!}(h)^2 + \frac{f'''(a)}{3!}(h)^3 + \dots}$$

Radius of Convergence (See previous Taylor Expansions)

a) $$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(2x)^{(2n+3)}}{(2n+3)!}}{\dfrac{(-1)^n(2x)^{(2n+1)}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2n+1)!\,(2x)^{(2n+3)}}{(2n+3)(-1)^n(2x)^{(2n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)(2x)^2}{(2n+3)(2n+2)}\right| = 0$$

$\boxed{\text{Radius of convergence for } \sin(2x) \text{ is } (-\infty,\infty)}$

b) $$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^n(x-1)^{n+1}}{(n+1)}}{\dfrac{(-1)^{n-1}(n-1)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(x-1)^{(n+1)}(n)}{(-1)^{(n+1)}(x-1)^n(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n(1-x)}{n+1}\right| = 1-x$$

$-1 < 1-x < 1$

$-2 < -x < 0$

$0 < x < 2$

$\boxed{\text{Radius of convergence for } \ln(2x) \text{ is } (0,2)}$

c) $$\lim_{n\to\infty}\left|\frac{\dfrac{2^{(\cancel{n+1})}e^2(x-1)^{(\cancel{n+1})}}{(n+1)\cancel{!}}}{\dfrac{2^{\cancel{n}}e^2(x-1)^{\cancel{n}}}{n\cancel{!}}}\right| = \lim_{n\to\infty}\left|\frac{2(x-1)}{(n+1)}\right| = 0$$

$\boxed{\text{Radius of convergence for } e^{2x} \text{ is } (-\infty,\infty)}$

d/e) Radius of convergence for $3x^2-2x+5$ is $(-\infty,\infty)$ when centered at $x_0=0$ and $x_0=1$.

f) The series converges where the denominator of $f(x)$ does not equal zero.

$(-\infty, -1) \cup (-1, \frac{5}{3}) \cup (\frac{5}{3}, \infty)$ is where the function converges.

g) $$\lim_{n\to\infty} \left| \frac{\frac{e^{-2} + (-1)^{n+1} e^{2}}{2\cdot(n+1)!}(x-1)^{n+1}}{\frac{e^{-2} + (-1)^{n} e^{2}}{2\cdot(n)!}(x-1)^{n}} \right| = \lim_{n\to\infty} \left| \frac{(e^{-2}+(-1)^{n+1}e^{2})\cancel{(2)}(\cancel{n!})(x-1)^{n+1}}{(e^{-2}+(-1)^{n}e^{2})\cancel{(2)}(n+1)!\,(\cancel{x-1})^{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(e^{-2}+(-1)^{n+1}e^{2})(x-1)}{(e^{-2}+(-1)^{n}e^{2})(n+1)} \right| = 0$$

$\boxed{\text{Radius of convergence for } \cosh(x-3) \text{ is } (-\infty, \infty).}$

h) $$\lim_{n\to\infty} \left| \frac{\frac{F^{n+1}(a)(x-a)^{n+1}}{(n+1)!}}{\frac{F^{n}(a)(x-a)^{n}}{n!}} \right| = \lim_{n\to\infty} \left| \frac{F^{n+1}(a)(x-a)}{F^{n}(a)(n+1)} \right| = \frac{F^{n+1}(a)(x-a)}{F^{n}(a)}$$

$$-1 < \frac{F^{n+1}(a)}{F^{n}(a)} x - \frac{F^{n+1}(a)}{F^{n}(a)} a < 1$$

$$\boxed{-\frac{F^{n}(a)}{F^{n+1}(a)} + a < x < \frac{F^{n}(a)}{F^{n+1}(a)} + a}$$

i) $$\lim_{n\to\infty}\left|\frac{\frac{F^{n+1}(x)(a-x)^{n+1}}{(n+1)!}}{\frac{F^{n}(x)(a-x)^{n+1}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{F^{n+1}(x)(a-x)}{F^{n}(x)(n+1)}\right|$$

$$= \frac{F^{n+1}(x)(a-x)}{F^{n}(x)} \qquad -1 < \frac{F^{n+1}(x)}{F^{n}(x)}a - \frac{F^{n+1}(x)}{F^{n}(x)}x < 1$$

$$\boxed{-\frac{F^{n}(x)}{F^{n+1}(x)} + x < a < \frac{F^{n}(x)}{F^{n+1}(x)} + x}$$

j) $$\lim_{n\to\infty}\left|\frac{\frac{F^{n+1}(a)(h)^{n+1}}{(n+1)!}}{\frac{F^{n}(a)(h)^{n}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{F^{n+1}(a)(h)}{F^{n}(a)(n+1)}\right|$$

$$= \frac{F^{n+1}(a)h}{F^{n}(a)} \qquad -1 < \frac{F^{n+1}(a)h}{F^{n}(a)} < 1$$

$$\boxed{-\frac{F^{n}(a)}{F^{n+1}(a)} < h < \frac{F^{n}(a)}{F^{n+1}(a)}}$$

Computing Antiderivatives

a) $\int x\sin(2x)\,dx$ $\quad u = x \quad dv = \sin(2x)$
$\quad du = dx \quad v = -\frac{1}{2}\cos(2x)$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$$

$$= \boxed{-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C}$$

b) $\int x e^{x^2}\,dx$ $\quad u = x^2$
$\quad du = 2x\,dx$

$$= \int \frac{1}{2}e^u\,du = \boxed{\frac{1}{2}e^{x^2} + C}$$

c) $\int x e^x\,dx$ $\quad u = x \quad dv = e^x$
$\quad du = dx \quad v = e^x$

$$= xe^x - \int e^x\,dx$$

$$= \boxed{xe^x - e^x + C}$$

d) $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ $\qquad \int e^{x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)n!} + C}$$

e) $\int x\sqrt{1+x}\,dx$ $\quad u = 1+x \quad x = u-1$
$\quad du = dx$

$$= \int (u-1)u^{1/2}\,du$$

$$= \int u^{3/2} - u^{1/2}\,du = \boxed{\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C}$$

f) $\int \sec(\theta)d\theta = \int \sec(\theta) \cdot \frac{\sec(\theta)+\tan(\theta)}{\sec(\theta)+\tan(\theta)} d\theta$

$= \int \frac{\sec^2(\theta)+\tan(\theta)\sec(\theta)}{\sec(\theta)+\tan(\theta)} d\theta$ $\quad u = \sec(\theta)+\tan(\theta)$

$du = \tan(\theta)\sec(\theta)+\sec^2(\theta)\,d\theta$

$= \int \frac{1}{u} du = \boxed{\ln|\sec(\theta)+\tan(\theta)| + C}$

g) $\int \sec^2(\theta)\,d\theta = \boxed{\tan(\theta)+C}$

h) $\int \operatorname{sech}^2(\theta)\,d\theta = \boxed{\tanh(\theta)+C}$

i) $\int \frac{x^2+2}{7-x^2}dx = -\int \frac{x^2+2}{x^2-7}dx = -\left(\int \frac{x^2-7}{x^2-7}dx + \int \frac{9}{x^2-7}dx\right)$

$= -x - \int \frac{9}{x^2-7}dx$ $\quad \frac{9}{x^2-7} = \frac{A}{x+\sqrt{7}} + \frac{B}{x-\sqrt{7}}$

$9 = (x-\sqrt{7})A + (x+\sqrt{7})B \qquad 9 = Ax + Bx - \sqrt{7}A + \sqrt{7}B$

$A+B=0$ $\quad B\sqrt{7}+B\sqrt{7}=9$ $\quad B=\frac{9}{2\sqrt{7}}$ $\quad A=-\frac{9}{2\sqrt{7}}$

$A=-B$ $\quad 2B\sqrt{7}=9$

$= -x + \frac{9}{2\sqrt{7}}\int \frac{dx}{x+\sqrt{7}} - \frac{9}{2\sqrt{7}}\int \frac{dx}{x-\sqrt{7}}$

$= \boxed{-x + \frac{9}{2\sqrt{7}}\left(\ln|x+\sqrt{7}| - \ln|x-\sqrt{7}|\right) + C}$

j) $\int \frac{1}{p(a-bp)}\,dp$

$$\frac{1}{p(a-bp)} = \frac{C}{p} + \frac{D}{a-bp}$$

$$1 = Ca - Cbp + Dp$$

$Ca = 1$ $-Cb + D = 0$

$C = \frac{1}{a}$ $-\frac{b}{a} + D = 0$

$D = \frac{b}{a}$

$$= \frac{1}{a}\int \frac{dp}{p} + \frac{b}{a}\int \frac{dp}{a-bp}$$

$$= \frac{1}{a}\ln|p| - \frac{1}{a}\ln|a-bp| + C$$

$$= \boxed{\frac{1}{a}\ln\left|\frac{p}{a-bp}\right| + C}$$

Simple Initial Value Problems

a) $\frac{dx}{dt} = 3x$ $\quad \frac{dx}{x} = 3\,dt$ $\quad \ln|x| = 3t + C$

$x_0 = 1$ $\quad x = Ae^{3t}$ $\quad 1 = Ae^{3(0)} \Rightarrow 1 = A$

$$\boxed{x = e^{3t}}$$

b) $\frac{dx}{dt}\ 3tx$ $\quad x_0 = 1$ $\quad \frac{dx}{x} = 3t\,dt$ $\quad \ln|x| = \frac{3}{2}t^2 + C$

$x = e^{\frac{3}{2}t} \cdot A$ $\quad 1 = (1)\cdot A$ $\quad \boxed{x = e^{3/2\,t}}$

c) $\frac{dx}{x(.1 - .003x)} = dt$ $\quad \frac{1}{x(.1-.003x)} = \frac{A}{x} + \frac{B}{.1-.003x}$

$1 = .1A - .003Ax + Bx$

$.1A = 1 \quad A = 10 \quad -.03 + B = 0$

$B = .03$

$$\Rightarrow 10\int \frac{dx}{x} + .03\int \frac{dx}{.1-.003x} = t + C$$

$$\Rightarrow 10\ln|x| - 10\ln|.1-.003x| = t + C$$

$\frac{x}{.1-.003x} = Ae^{t/10}$ $\quad x(0) = 4$ $\quad \frac{4}{.1-.012} = A = \frac{500}{11}$

$$x = (.1)\frac{500}{11}e^{t/10} - (.003)\frac{500}{11}e^{t/10}x$$

$$\boxed{x = \frac{\frac{50}{11}e^{t/10}}{1 + \frac{1.5}{11}e^{t/10}}}$$

d) Using previous work. $\dfrac{x}{.1 - .003x} = Ae^{t/10}$

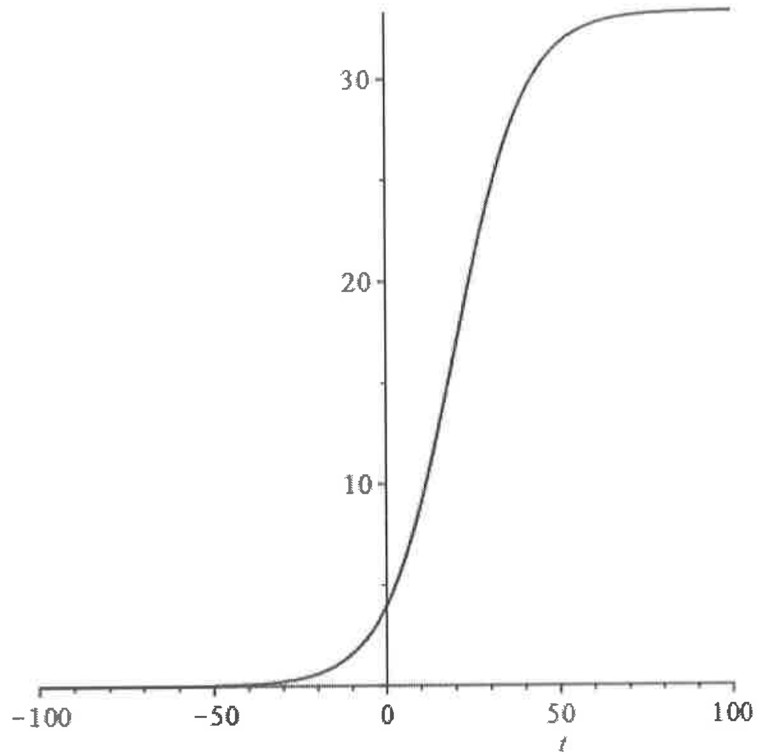
now $x(0) = 400$

$$\frac{400}{.1 - .003(400)} = A \qquad \frac{400}{-1.1} = A = \frac{-4000}{11}$$

$$x = (.1)\frac{-4000}{11}e^{t/10} - (.003)\frac{-4000}{11}e^{t/10}x$$

$$\boxed{x = \frac{\frac{400}{11}e^{t/10}}{1 + \frac{12}{11}e^{t/10}}}$$

$\frac{dx}{dt} = 0.1x - 0.003x^2$ where $x(0) = 4$



$\frac{dx}{dt} = 0.1x - 0.003x^2$ where $x(0) = 400$